ON THE ACCURACY OF THE FINITE-DIFFERENCE METHOD USING MESH GRADING

Paolo Mezzanotte, Luca Roselli
University of Perugia, Perugia / Italy

Christian Huber, Horst Zscheile, and Wolfgang Heinrich
Ferdinand-Braun-Institut, Berlin / Germany

ABSTRACT

The coarseness error of the Finite-Difference method is studied using a typical planar waveguide as a reference. Because of the field singularities involved a graded mesh proves to be superior compared to the equidistant case. A grading strategy with optimum efficiency is presented.

MOTIVATION

Nowadays numerical methods for electromagnetic simulation constitute an indispensable tool for solving microwave engineering problems. Among the different approaches, the Finite-Difference method in time domain (FDTD) has received great attention due to its flexibility and its direct relationship with Maxwell's equations. Commonly, discretization follows the central difference scheme according to Yee [1]. As well known, this scheme exhibits second-order accuracy as long as an equidistant mesh is used. In the case of mesh grading, this characteristic deteriorates to the first order. In the past, several approaches were proposed to overcome this limitation (e.g. [2]). This is accomplished, however, at the expense of other properties such as flexibility.

In the discussions on this topic, however, one fact needs to be emphasized that appears to be not as generally known as the above-mentioned ones: In the derivation of the second-order accuracy behavior, one assumes regular, i.e. bounded, fields. If the discretized domain contains field singularities, the order of accuracy is determined by the singularity rather than by the inherent order of accuracy (see, e.g., Finite-Element method [3]). Such singularities occur at each metallic corner. In the case of planar microwave circuits, for instance, the field behavior near the corners or edges dominates the overall behavior. Hence, it becomes questionable whether the second-order rule provides a good estimation for practical applications.

In this context, the paper contributes results on two aspects:

- The accuracy of the FD method in the presence of field singularities is investigated.
- Information is provided how to choose discretization in order to optimize the tradeoff between accuracy and numerical efforts.

One should note that our considerations focus on the so-called coarseness error, i.e., the error caused by the limited spatial resolution. There are, of course, other sources of error, e.g. the dispersion due to discretization, which, however, are beyond the scope of this paper.

METHOD OF ANALYSIS

In order to determine the accuracy one needs to treat a structure for which the results are analytically known or can be derived by other highly accurate methods. On this reason, we choose the waveguide problem depicted in Fig. 1.

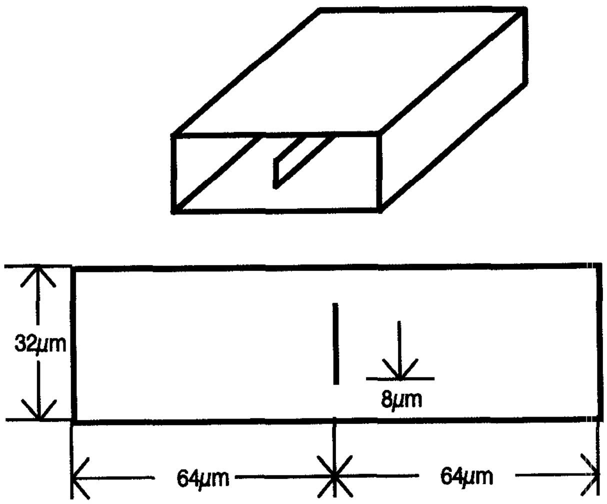

Fig. 1: The waveguide under consideration and its cross-section (center conductor and enclosure are ideally conducting).

CH3577-4/95/0000-0781$01.00 © 1995 IEEE

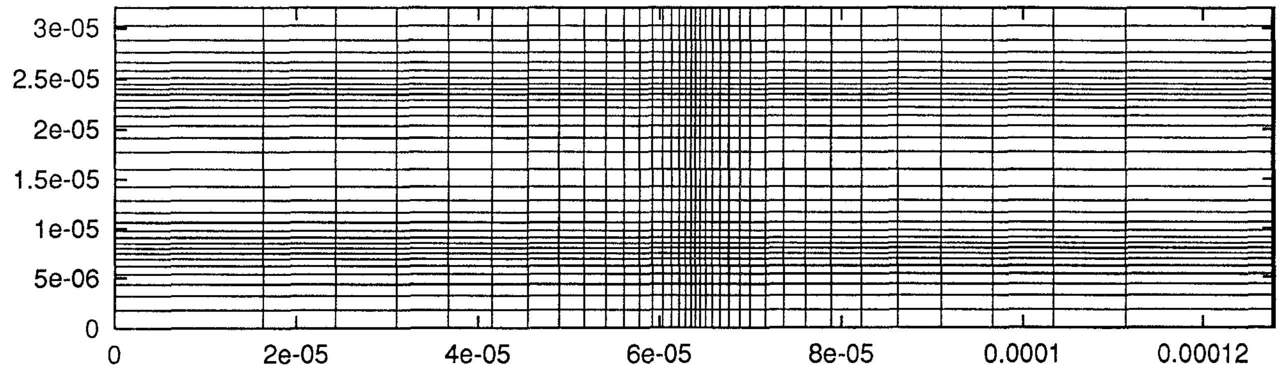

Fig. 2: Example for discretization by a graded mesh (smallest cell size $\Delta s_{min} = 0.5\mu$m, grading according to a geometric series with factor $q = \sqrt[4]{2}$, all dimensions in meter).

With respect to the intended accuracy investigations, the structure offers the following desirable properties:

- The fields concentrate in the slots. Thus, it represents a good example for the situation in planar circuits where the field behavior in a slot region determines the characteristics (e.g.: CPW, slotline, coupled microstrips).
- The geometry is symmetrical and relatively simple. Therefore, the influence of the field singularity can be separated and it is not clouded by other effects.
- The waveguide supports a pure TEM fundamental wave. Thus, the characteristic impedance Z is well defined and may be used as an indicator for overall accuracy.
- Assuming the lateral walls to be removed the characteristic impedance of the TEM mode can be analytically derived by conformal mapping.

The dimensions are chosen so that the slot geometry corresponds to the MMIC-typical situation and that the cut-off frequencies for the higher-order modes are sufficiently high.

For analysis, we employ Finite-Difference methods both in time (FDTD [4]) and frequency domain (FDFD [5]). In the FDTD case, a three-dimensional treatment is applied exciting the structure with a Gaussian pulse, whereas in the FDFD case a two-dimensional eigenvalue problem is solved. Comparing the results of both methods we found that the deviations are of minor importance and do not affect the following investigations.

MESHING STRATEGY

In order to separate the influence of the different mesh parameters we proceed as follows:

(i) An equidistant mesh is used starting with a cell size of $\Delta s = \Delta x = \Delta y = 8\mu$m. Subsequently, its value is reduced to 4μm, 2μm, etc.

(ii) A graded mesh is applied with the smallest cell size Δs_{min} located at the corners of the infinitely thin inner conductor. Starting from these points, the cell size is increased successively by a constant factor q. Hence, the cell sizes follow a geometric series. Fig. 2 illustrates this strategy.

RESULTS

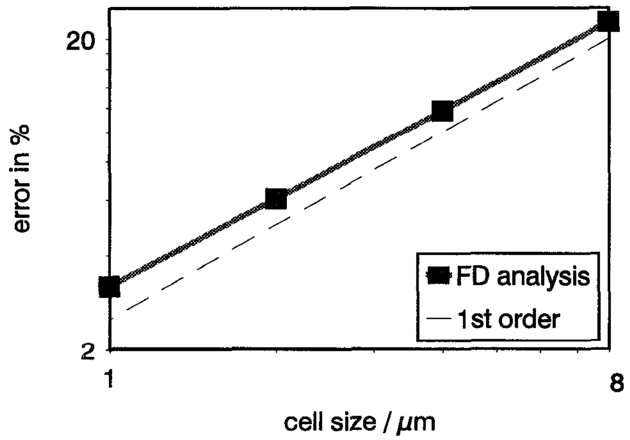

Fig. 3: Percentage error in characteristic impedance Z of the waveguide according to Fig. 1 versus cell size for an equidistant mesh (the relative error refers to the analytical value $Z = 94.2\Omega$ obtained by conformal mapping).

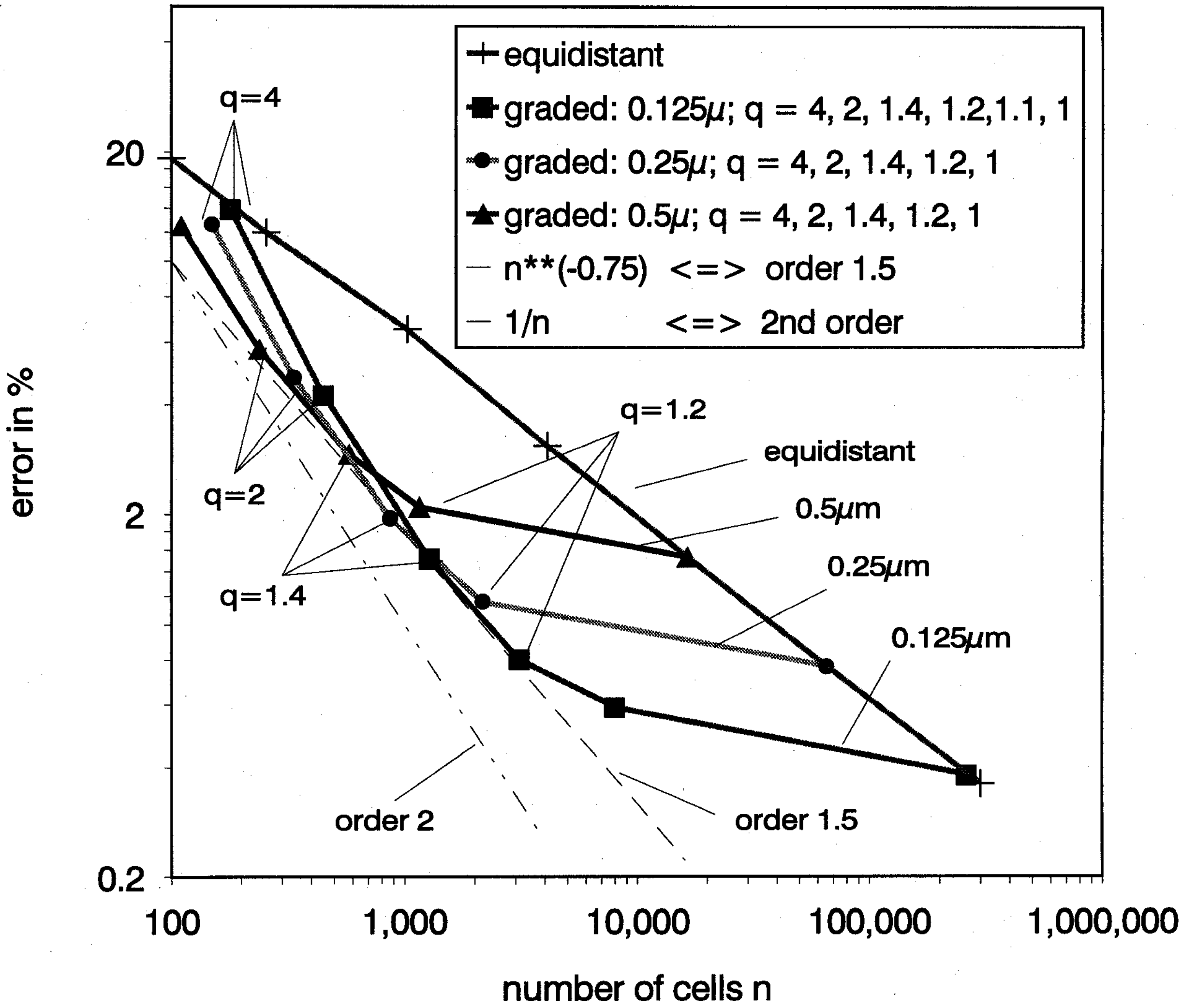

Fig. 4: Error of characteristic impedance Z against number of cells n required: comparison between equidistant and graded mesh with $\Delta s_{min} = 0.5\mu$m, 0.25μm, and 0.125μm and the grading factor q varied as given in the inset (other data as in Fig. 3).

First, the equidistant case will be considered. In Fig. 3, the error is plotted as a function of the cell size. Clearly, one observes a 1st-order behavior. At the first glance, this may surprise since one expects a 2nd-order characteristic.

The discrepancy is caused by the field singularity. The investigated structure with an infinitely thin strip exhibits an edge singularity of the order 0.5 (i.e., $E \sim 1/\sqrt{r}$ with r denoting the distance from the edge). Incorporating this behavior into the Finite-Difference equations, one derives for the cells at the edge an error order worse than that of the regular case. More precisely, a rule $\Delta E \sim \Delta s^{0.5}$ is found. Presumably, certain errors cancel out when calculating the impedance Z from the fields, which results in a first-order characteristic.

Second, a graded mesh is applied (see Fig. 2) and the influence of both the smallest cell size Δs_{min} and the grading factor q is studied. Fig. 4 illustrates the results. It presents curves varying q with Δs_{min} kept constant. The error in characteristic impedance Z is plotted against the number n of cells in the cross-section, which corresponds to the computer efforts involved. Therefore, the diagram provides information on an application-oriented figure of merit, that is which accuracy can be achieved by a given number of cells, or which is the numerical expense for a given accuracy. In other words, the nearer a curve to the origin of the diagram the more effective the discretization.

The results demonstrate clearly that mesh grading leads to considerable improvement in efficiency. It outperforms the equidistant case even for q values as large as 4. Regarding the corresponding order of accuracy, a value of about 1.5 is achieved compared to 1 for the equidistant mesh. The curves in Fig. 4 indicate that there is an optimum choice for the grading factor q. Independent of the smallest cell size, a value q in the range 1.2...2 yields the best results in terms of efficiency. For larger values of q, accuracy degrades due to poor resolution. For $q \rightarrow 1$, on the other hand, the increase in mesh size does not lead to an equivalent improvement in accuracy (see Fig. 5).

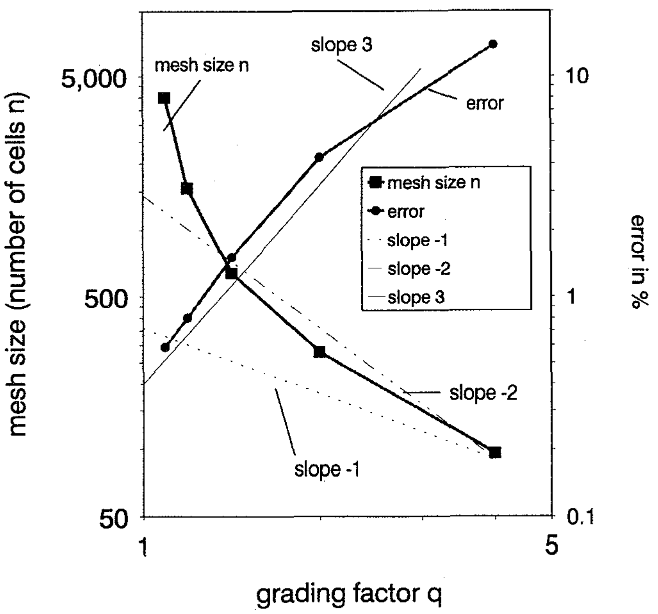

Fig. 5: Mesh size n and error in Z against grading factor q for $\Delta s_{min} = 0.125\mu$m (other data identical to Fig. 3).

CONCLUSIONS

From the results presented the following conclusions with regard to the FD analysis of planar circuits can be drawn:

- The well-known second-order error behavior of the FD method refers only to regular fields and does not hold at field singularities. For the infinitely thin strip, we find a only a first-order behavior in characteristic impedance. That means: The overall accuracy is determined primarily by the spatial resolution at the metallic edges and corners.
- Although the introduction of mesh grading increases the global FD error to the first order, it yields a much better accuracy than the equidistant version for a given mesh size. This is due to the improved field resolution near the singularities.
- If one uses a graded mesh with a constant ratio q relating the neighbouring discretization steps, one has two degrees of freedom: the smallest cell size Δs_{min} and the grading factor q. Our investigations indicate that choosing q in the range 1.2...2 yields optimum efficiency.

ACKNOWLEDGEMENTS

The authors are grateful to Dr. Klaus Beilenhoff from the Technical University of Darmstadt for helpful discussions and for providing FDFD results.

REFERENCES

[1] K.S. Yee, "Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media," *IEEE Trans. Antenna Propagat.*, Vol. AP-14, pp. 302-307, May 1966.

[2] S. Xiao and R. Vahldieck, "An improved 2D-FDTD algorithm for hybrid mode analysis of quasi-planar transmission lines," *1993 International Microwave Symposium Digest*, Vol. 1, pp. 421-424.

[3] J.M. Gil and J. Zapata, "Efficient singular element for finite element analysis of quasi-TEM transmission lines and waveguides with sharp metal edges," *IEEE Trans. Microwave Theory Tech.*, Vol. MTT-42, pp. 92-98, Jan 1994.

[4] P. Mezzanotte, L. Roselli, and R. Sorrentino, "Simulation of interconnection and package interaction phenomena in MMIC's by FDTD," *Proc. of the 2nd Topical Meeting on Electrical Performance of Electronic Packaging*, pp. 139-142, Oct 1993.

[5] K. Beilenhoff, W. Heinrich, and H.L. Hartnagel, "Improved Finite-Difference formulation in the frequency domain for three-dimensional scattering problems," *IEEE Trans. Microwave Theory Tech.*, Vol. MTT-40, pp. 540-546, March 1992.

[6] L. Roselli, B. Bader, and W. Heinrich, "Comparison of Finite-Difference and Transmission-Line Matrix methods used for S-parameter analysis," *Proc. 1994 MIKON, Ksiaz, Poland*, Vol. 2, pp. 597-605.